Surds, Indices, and Logarithms

Radical

Definition of the Radical

For all real $x,\ y > 0$, and all integers $a > 0$,

$$\sqrt[a]{x} = y \text{ if and only if } x = y^a$$

where a is the **index**
$\sqrt{\ }$ is the **radical**
x is the **radicand**.

Surds

A number which can be expressed as a fraction of integers (assuming the denominator is never 0) is called a **rational number**. Examples of rational numbers are $\frac{5}{2}$, $-\frac{4}{5}$ and 2.

A number which cannot be expressed as a fraction of two integers is called an irrational number. Examples of irrational numbers are $\sqrt{2}$, $\sqrt[3]{7}$ and π.

An irrational number involving a root is called a surd. Surds occur frequently in trigonometry, calculus and coordinate geometry. Usually, the exact value of a surd cannot be determined but an approximate value of it can be found by using calculators or mathematical tables. In this chapter, $\sqrt{a}$ means the **positive** square root of *a* while $-\sqrt{a}$ means the **negative** square root of *a*.

General Rules of Surds

Multiplication of surds

$\sqrt{a} \times \sqrt{b} = \sqrt{a \times b}$

For example (i) $\sqrt{3} \times \sqrt{12} = \sqrt{3 \times 12} = \sqrt{36} = 6$
(ii) $\sqrt{32} \times \sqrt{2} = \sqrt{32 \times 2} = \sqrt{64} = 8$
(iii) $\sqrt{5} \times \sqrt{5} = \sqrt{5 \times 5} = \sqrt{25} = 5$

Division of surds

$$\sqrt{a} \div \sqrt{b} = \frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$$

For example (i) $\sqrt{72} \div \sqrt{2} = \sqrt{\frac{72}{2}} = \sqrt{36} = 6$

(ii) $\sqrt{45} \div \sqrt{5} = \sqrt{\frac{45}{5}} = \sqrt{9} = 3$

These rules are useful for simplifying two or more surds of for combining them into one single surd.

Note, however, that $\sqrt{3}+\sqrt{6} \neq \sqrt{3+6}$ and $\sqrt{6}-\sqrt{2} \neq \sqrt{6-2}$ which can be easily checked by a calculator; and, therefore, in general $\sqrt{a}+\sqrt{b} \neq \sqrt{a+b}$ and $\sqrt{a}-\sqrt{b} \neq \sqrt{a-b}$.

Example 1

(i) $3\sqrt{5}+\sqrt{5} = (3+1)\sqrt{5}$
$= 4\sqrt{5}$

(ii) $\sqrt{40} = \sqrt{4\times10} = \sqrt{4}\times\sqrt{10}$
$= 2\sqrt{10}$

Example 2

Simplify (i) $\sqrt{243}-\sqrt{12}+2\sqrt{75}$

(ii) $\sqrt{50}+\sqrt{8}+\sqrt{32}$

Solution:

(i)
$$\begin{aligned}\sqrt{243}-\sqrt{12}+2\sqrt{75} &= \sqrt{81\times3}-\sqrt{4\times3}+2\sqrt{25\times3}\\ &= \sqrt{81}\times\sqrt{3}-\sqrt{4}\times\sqrt{3}+2\sqrt{25}\times\sqrt{3}\\ &= 9\sqrt{3}-2\times\sqrt{3}+10\times\sqrt{3}\\ &= (9-2+10)\sqrt{3}\\ &= 17\sqrt{3}\end{aligned}$$

(ii)
$$\begin{aligned}\sqrt{50}+\sqrt{8}+\sqrt{32} &= \sqrt{25\times2}+\sqrt{4\times2}+\sqrt{16\times2}\\ &= \sqrt{25}\times\sqrt{2}+\sqrt{4}\times\sqrt{2}+\sqrt{16}\times\sqrt{2}\\ &= 5\times\sqrt{2}+2\times\sqrt{2}+4\times\sqrt{2}\\ &= (5+2+4)\sqrt{2}\\ &= 11\sqrt{2}\end{aligned}$$

Try This 1

Simplify (i) $\sqrt{27}$ (ii) $\sqrt{28}-\sqrt{175}+\sqrt{112}$ (iii) $\sqrt{5}\times\sqrt{125}\times\sqrt{8}$

Rationalization of the Denominator

When a fraction has a surd in its denominator, e.g. $\frac{3}{\sqrt{2}}$, it is usual to eliminate the surd in the denominator. In fact, the writing of surds in the denominators of fractions should be avoided. The process of removing this surd is called **rationalizing of the denominator**.

$\sqrt{m}+\sqrt{n}$ and $\sqrt{m}-\sqrt{n}$ are specially related surds known as **conjugate surds**. The product of conjugate surds is always a rational number.

$$(\sqrt{m}+\sqrt{n})(\sqrt{m}-\sqrt{n})=(\sqrt{m})^2-(\sqrt{n})^2=m-n$$

For example $(\sqrt{9}+\sqrt{5})(\sqrt{9}-\sqrt{5})=(\sqrt{9})^2-(\sqrt{5})^2=9-5=4$
$(\sqrt{7}+\sqrt{3})(\sqrt{7}-\sqrt{3})=(\sqrt{7})^2-(\sqrt{3})^2=7-3=4$

Example 3

Simplify $\frac{5}{\sqrt{3}}$.

Solution:

$$\frac{5}{\sqrt{3}}=\frac{5}{\sqrt{3}}\times\frac{\sqrt{3}}{\sqrt{3}}$$
$$=\frac{5\sqrt{3}}{3}$$

Example 4

Simplify $\frac{4}{\sqrt{7}+\sqrt{3}}$.

Solution:

$$\frac{4}{\sqrt{7}+\sqrt{3}}=\frac{4}{\sqrt{7}+\sqrt{3}}\times\frac{\sqrt{7}-\sqrt{3}}{\sqrt{7}-\sqrt{3}}$$
$$=\frac{4(\sqrt{7}-\sqrt{3})}{7-3}$$
$$=\sqrt{7}-\sqrt{3}$$

Example 5

Simplify, without using tables or calculators, the value of $\frac{1}{3-\sqrt{2}}+\frac{1}{3+\sqrt{2}}$.

Solution:

$$\frac{1}{3-\sqrt{2}}+\frac{1}{3+\sqrt{2}}=\frac{(3+\sqrt{2})}{(3-\sqrt{2})(3+\sqrt{2})}+\frac{(3-\sqrt{2})}{(3+\sqrt{2})(3-\sqrt{2})}$$
$$=\frac{(3+\sqrt{2})+(3-\sqrt{2})}{9-2}$$
$$=\frac{6}{7}$$

Try This 2

Simplify (i) $\frac{3}{\sqrt{12}}$ (ii) $\frac{1}{\sqrt{3}+\sqrt{7}}$ (iii) $\frac{\sqrt{7}+2}{\sqrt{7}-2}$

Answers to *Try This*

1. (i) $3\sqrt{3}$
   (ii) $\sqrt{7}$
   (iii) $50\sqrt{2}$

2. (i) $\frac{\sqrt{3}}{2}$
   (ii) $\frac{\sqrt{7}-\sqrt{3}}{4}$
   (iii) $\frac{11+4\sqrt{7}}{3}$

Indices

If a positive integer *a* is multiplied by itself three times, we get a^3, i.e. $a \times a \times a = a^3$. Here *a* is called the **base** and 3, the **index** or **power**. Thus a^4 means the 4th power of *a*.

In general, a^n means the *n*th power of *a*, where *n* is any positive index of the positive integer *a*.

Rules of Indices

There are several important rules to remember when dealing with indices.

If *a*, *b*, *m* and *n* are positive integers, then

(1) $a^m \times a^n = a^{m+n}$ e.g. $3^5 \times 3^8 = 3^{13}$

(2) $a^m \div a^n = a^{m-n}$ e.g. $5^{14} \div 5^3 = 5^{11}$

(3) $(a^m)^n = a^{mn}$ e.g. $(5^2)^6 = 5^{12}$

(4) $a^m \times b^m = (a \times b)^m$ e.g. $3^5 \times 2^5 = 6^5$

(5) $a^m \div b^m = \left(\frac{a}{b}\right)^m$ e.g. $5^4 \div 3^4 = \left(\frac{5}{3}\right)^4$

(6) $a^0 = 1$ e.g. $5^0 = 1$

(7) $a^{-n} = \frac{1}{a^n}$ e.g. $5^{-3} = \frac{1}{5^3}$

(8) $a^{\frac{1}{n}} = \sqrt[n]{a}$ e.g. $8^{\frac{1}{3}} = \sqrt[3]{8}$

(9) $a^{\frac{m}{n}} = \sqrt[n]{a^m} = (\sqrt[n]{a})^m$ e.g. $8^{\frac{2}{3}} = \sqrt[3]{8^2} = (\sqrt[3]{8})^2$

Example 1

Evaluate (i) 2^{-3} (ii) $8^{\frac{1}{3}}$ (iii) $16^{\frac{3}{4}}$ (iv) $25^{-\frac{3}{2}}$

Solution:

(i) $2^{-3} = \frac{1}{2^3} = \frac{1}{8}$

(ii) $8^{\frac{1}{3}} = \sqrt[3]{8} = 2$

(iii) $16^{\frac{3}{4}} = (\sqrt[4]{16})^3$

$= 2^3$

$= 8$

(iv) $25^{-\frac{3}{2}} = \frac{1}{25^{\frac{3}{2}}}$

$= \frac{1}{(\sqrt{25})^3}$

$= \frac{1}{5^3} = \frac{1}{125}$

Try This 1

Evaluate each of the following without using a calculator

(i) 7^{-1} (ii) 17^0 (iii) $49^{\frac{3}{2}}$ (iv) $8^{-\frac{2}{3}}$

(v) $243^{\frac{3}{5}}$ (vi) $81^{\frac{1}{4}}$ (vii) $\left(\frac{1}{27}\right)^{-\frac{4}{3}}$ (viii) $\left(\frac{1}{4}\right)^{-2}$

Example 2

Simplify (i) $a^{\frac{1}{3}} \times a^{\frac{2}{5}} \div a^{\frac{1}{2}}$ (ii) $(a^3b^2)^4$ (iii) $\sqrt[3]{a} \div \sqrt[5]{a^2} \times (a^{-1})^{\frac{1}{2}}$

Solution:

(i) $a^{\frac{1}{3}} \times a^{\frac{2}{5}} \div a^{\frac{1}{2}}$

$= a^{\frac{1}{3}+\frac{2}{5}-\frac{1}{2}}$

$= a^{\frac{7}{30}}$

(ii) $(a^3b^2)^4$

$= a^{3\times 4}b^{2\times 4}$

$= a^{12}b^8$

(iii) $\sqrt[3]{a} \div \sqrt[5]{a^2} \times (a^{-1})^{\frac{1}{2}}$

$= a^{\frac{1}{3}} \div a^{\frac{2}{5}} \times a^{-\frac{1}{2}}$

$= a^{\frac{1}{3}} \div a^{\frac{2}{5}} \times a^{-\frac{1}{2}}$

$= a^{\frac{1}{3}-\frac{2}{5}+\left(-\frac{1}{2}\right)}$

$= a^{-\frac{17}{30}}$

Try This 2

Simplify each of the following, giving your answer in index form:

(i) $a^3 \div a^{-4} \times a^2$ (ii) $16a^{-\frac{5}{2}} \div 4a^{-\frac{3}{2}}$ (iii) $(a^{\frac{1}{3}} \times b^{\frac{2}{5}})^{15}$

Solving Exponential Equations

Example 3

Solve the following exponential equations

(i) $2^x = 32$ (ii) $4^{x+1} = 0.25$

Solution:

(i)

$2^x = 32$

$2^x = 2^5$

$\therefore x = 5$

(ii)

$4^{x+1} = 0.25$

$4^{x+1} = \frac{1}{4}$

$4^{x+1} = 4^{-1}$

$x+1 = -1$

$\therefore x = -2$

> ***Try This 3***
>
> Solve the following equations:
>
> (i) $3^x = 81$ (ii) $32^x = 8$ (iii) $7^x = \frac{1}{49}$
>
> (iv) $5^x = 1$ (v) $3^{4x} = 27^{x+3}$ (vi) $4^x \times 3^{2x} = 6$

Example 4

Solve the equation $2^{2x+3} + 2^{x+3} = 1 + 2^x$.

Solution:

$2^{2x+3} + 2^{x+3} = 1 + 2^x$

$2^x \times 2^x \times 2^3 + 2^x \times 2^3 = 1 + 2^x$

Let $y = 2^x$

$8y^2 + 8y = 1 + y$

$8y^2 + 7y - 1 = 0$

$(8y-1)(y+1) = 0$

$y = \frac{1}{8}$ or -1

When $y = \frac{1}{8}$

$2^x = \frac{1}{8}$

$2^x = 2^{-3}$

$\therefore x = -3$

when $y = -1$

$2^x = -1$ (inadmissible)

Try This 4
Solve the equation $3^{2x+1} + 9 = 3^{x+3} + 3^x$.

Example 5
If $3^x \times 9^{2y} = 27$ and $2^x \times 4^{-y} = \frac{1}{8}$, calculate the values of *x* and *y*.

Solution:
$3^x \times 9^{2y} = 27 \quad \cdots\cdots\cdots(1)$

$2^x \times 4^{-y} = \frac{1}{8} \quad \cdots\cdots\cdots(2)$

From (1):
$$3^x \times (3^2)^{2y} = 3^3$$
$$3^x \times 3^{4y} = 3^3$$
$$3^{x+4y} = 3^3$$
$$x + 4y = 3 \quad \cdots\cdots(3)$$

From (2):
$$2^x \times (2^2)^{-y} = \frac{1}{2^3}$$
$$2^x \times 2^{-2y} = 2^{-3}$$
$$2^{x-2y} = 2^{-3}$$
$$x - 2y = -3 \quad \cdots\cdots(4)$$

$(3)-(4)$:
$$6y = 6$$
$$y = 1$$

Substitute $y = 1$ into (3):
$$x + 4(1) = 3$$
$$x = -1$$

$\therefore x = -1$ and $y = 1$.

Try This 5
Solve the simultaneous equations $3^{x+y} = 243$, $2^{2x-5y} = 8$

Answers to *Try This*

1.	(i)	$\frac{1}{7}$	(ii)	1	(iii)	343	(iv)	$\frac{1}{4}$
	(v)	27	(vi)	3	(vii)	81	(viii)	16
2.	(i)	a^9	(ii)	$\frac{4}{a}$	(iii)	a^5b^6		
3.	(i)	$x=4$	(ii)	$x=\frac{3}{5}$	(iii)	$x=-2$		
	(iv)	$x=0$	(v)	$x=9$	(vi)	$x=\frac{1}{2}$		

4. $x=-1$ or 2

5. $x=4,\ y=1$

Logarithms

Definition:
For any number y such that $y = a^x$ ($a > 0$ and $a \neq 1$), the logarithm of y to the base a is defined to be x and is denoted by $\log_a y$.

Thus if $y = a^x$, then $\log_a y = x$

For example, $81 = 3^4$ $\quad \therefore \log_3 81 = 4$

$100 = 10^2$ $\quad \therefore \log_{10} 100 = 2$

Note: The logarithm of 1 to any base is 0, i.e. $\log_a 1 = 0$.
The logarithm of a number to a base of the same number is 1, i.e. $\log_a a = 1$.
The logarithm of a negative number is undefined.

Example 1
Find the value of (i) $\log_2 64$ (ii) $\log_9 3$

(iii) $\log_3 \frac{1}{9}$ (iv) $\log_8 0.25$

Solution:
(i) Let $\log_2 64 = x$

$$64 = 2^x$$
$$2^6 = 2^x$$
$$\therefore x = 6$$

(ii) Let $\log_9 3 = x$

$$3 = 9^x$$
$$3 = 3^{2x}$$
$$1 = 2x$$
$$\therefore x = \frac{1}{2}$$

(iii) Let $\log_3 \frac{1}{9} = x$

$$\frac{1}{9} = 3^x$$
$$3^{-2} = 3^x$$
$$\therefore x = -2$$

(iv) Let $\log_8 0.25 = x$

$$0.25 = 8^x$$
$$\frac{1}{4} = 2^{3x}$$
$$2^{-2} = 2^{3x}$$
$$3x = -2$$
$$\therefore x = -\frac{2}{3}$$

Laws of Logarithms

(1) $\log_a mn = \log_a m + \log_a n$ e.g. $\log_3 5 + \log_3 2 = \log_3 10$

(2) $\log_a \frac{m}{n} = \log_a m - \log_a n$ e.g. $\log_3 5 + \log_3 4 = \log_3\left(\frac{5}{4}\right)$

(3) $\log_a m^p = p\log_a m$ e.g. $\log_{10} 5^2 = 2\log_{10} 5$

Example 2

Without using tables, evaluate $\log_{10}\frac{41}{35} + \log_{10} 70 - \log_{10}\frac{41}{2} + 2\log_{10} 5$.

Solution:

$$\log_{10}\frac{41}{35} + \log_{10} 70 - \log_{10}\frac{41}{2} + 2\log_{10} 5$$
$$= \log_{10}\left(\frac{41}{35}\times 70 \div \frac{41}{2}\times 5^2\right)$$
$$= \log_{10} 100$$
$$= \log_{10} 10^2$$
$$= 2\log_{10} 10$$
$$= 2$$

Try This 1

Simplify $2\log_3 5 - \log_3 10 + 3\log_3 4$.

Changing the Base of Logarithms

Logarithms to base 10 such as $\log_{10} 5$ and $\log_{10}(x+1)$ are called **common logarithms**. An alternative form of writing $\log_{10} 5$ is $\lg 5$ .Common logarithms can be evaluated using a scientific calculator.

Logarithms to base *e* such as $\log_e 3$ and $\log_e x$ are called **Natural logarithms** or **Napierian logarithms**. Natural logarithms are usually written in an alternative form, for example, $\log_e 3$ is written as $\ln 3$. (*Note*: $e = 2.718...$)

If *a*, *b*, and *c* are positive numbers and $a \neq 1$, then $\log_a b = \frac{\log_c b}{\log_c a}$.

Example 3
Find the value of $\log_5 16$.

Solution:
$$\log_5 16 = \frac{\log_{10} 16}{\log_{10} 5} = \frac{1.204}{0.699} = 1.722$$

> ***Try This 2***
> Find the value of $\log_4 54$.

Solving Logarithmic Equations

Example 4
Solve the equation $3^x = 18$.

Solution:
$3^x = 18$
Taking logarithms to base 10 on both sides,
$$\log_{10} 3^x = \log_{10} 18$$
$$x \log_{10} 3 = \log_{10} 18$$
$$x = \frac{\log_{10} 18}{\log_{10} 3} = \frac{1.2553}{0.4771}$$
$$= 2.631$$

> ***Try This 3***
> Solve the equation $5^{x+1} = 30$.

Example 5
Given that $\log_{10} 4 + 2\log_{10} p = 2$, calculate the value of *p* without using tables or calculators.

Solution:
$$\log_{10} 4 + 2\log_{10} p = 2$$
$$\log_{10}(4 \times p^2) = 2$$
$$4p^2 = 10^2$$
$$p^2 = \frac{100}{4}$$
$$p^2 = 25$$
$$p = \pm 5$$
Since *p* cannot be -5 because $\log_{10}(-5)$ is not defined, $p = 5$.

Try This 4

Solve the equation $\log_2 \frac{3x+1}{2x-7} = 3$.

Example 6

Solve the equation $\log_{10}(3x+2) - 2\log_{10} x = 1 - \log_{10}(5x-3)$.

Solution:

$$\log_{10}(3x+2) - 2\log_{10} x = 1 - \log_{10}(5x-3)$$

$$\log_{10}(3x+2) - \log_{10} x^2 + \log_{10}(5x-3) = 1$$

$$\log_{10} \frac{(3x+2)(5x-3)}{x^2} = 1$$

$$\frac{(3x+2)(5x-3)}{x^2} = 10^1$$

$$15x^2 - 9x + 10x - 6 = 10x^2$$

$$5x^2 + x - 6 = 0$$

$$(5x+6)(x-1) = 0$$

$$\therefore x = -\frac{6}{5} \text{ or } x = 1$$

Since x cannot be negative, $x = 1$.

Try This 5

Solve the equation $\log_2 x^2 = 4 + \log_2(x-3)$.

Answers to *Try This*

1. $\log_3 160$
2. 2.877
3. $x = 1.113$
4. $x = \frac{57}{13}$
5. $x = 4$ or 12.